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A STABLE HIGH-ORDER METHOD FOR THE HEATED CAVITY PROBLEM

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SUMMARY

A fourth-order method, without using extrapolation, is developed for the steady-state solution of a non-linear system of three simultaneous partial differential equations for the flow of a fluid in a heated closed cavity. The method is a finite difference method which has converged for all Rayleigh numbers *Ra* of physical interest and all Prandtl numbers *Pr* attempted. The results are presented and compared with some of the accurate results available in de Vahl Davis and Jones, Shay and Schultz, and Dennis and Hudson. The method used to develop the fourth-order method presented in this paper can be used to develop high-order methods for other partial differential equations. The method was developed to be stable without using the upwinding technique.

KEY WORDS A fourth-order method Navier–Stokes equation Stability Finite difference

INTRODUCTION

The purpose of this paper is to develop a fourth-order finite difference method to solve the Navier–Stokes equations describing the flow of a fluid across the square between two plane parallel straight line boundaries. The problem is formulated in terms of flow in a rectangular cavity in which the top wall has a temperature T_1, and the bottom wall a temperature T_0, with $T_1 > T_0$.

The problem to be considered is formulated as follows. Let Ω be a square region $(0, 1) \times (0, 1)$, with vertices A, B, C, D, as placed in Figure 1.

On Ω the equations of motion to be satisfied are as follows:

$$\Delta\psi = -\omega, \tag{1}$$

$$\Delta T + \psi_x T_y - \psi_y T_x = 0, \tag{2}$$

$$\Delta\omega + (1/Pr)(\psi_x \omega_y - \psi_y \omega_x) + Ra\, T_y = 0, \tag{3}$$

where ψ, T and ω represent the stream, temperature and vorticity functions. Equations (2) and (3) are the non-conservation form. For the case where the surfaces between the hot and cold walls are insulated, the boundary conditions to be satisfied are

$$\psi = 0 \quad \text{on ABCDA}, \tag{4}$$

$$\psi_y = 0, \quad T = 0 \quad \text{on AB}, \tag{5}$$

$$\psi_x = 0, \quad T_x = 0 \quad \text{on AD and BC}, \tag{6}$$

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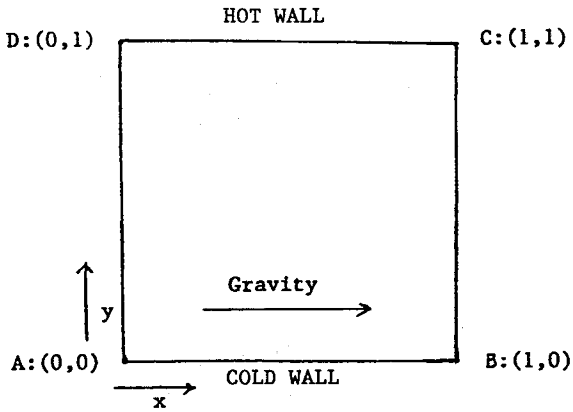

Figure 1

$$\psi_y = 0, \quad T = 1 \quad \text{on CD.} \tag{7}$$

For the case where the temperature varies linearly along the walls separating the hot and cold surfaces, the condition $T_x = 0$ is replaced by $T = y$ in equation (6).

In general, these equations cannot be solved by analytical means. Experimental work has been done by Mull and Reicher,[2] Elder,[3] Torrance *et al.*[4] and Eckert and Carlson.[5] Numerical results have been obtained by many authors, including Poots,[6] Rosen,[7] Elder,[3] de Vahl Davis,[8] Wilkes and Churchill,[9] Rubel and Landis,[10] Newell and Schmidt,[11] Schultz [1] and Shay and Schultz.[12] However, many of the results were obtained for limited values of Ra and Pr, or with lower-order approximations. Poots[6] used a series expansion, while Rosen[7] used linear programming techniques and Newell and Schmidt[11] used central difference approximations to first derivative terms. They all used only a value of $Pr = 0{\cdot}73$. Poots and Rosen could not obtain convergence for $Ra > 10\,000$. Others[8,3,9,10] also used central difference approximations, which tend to be unstable for small values of Pr. De Vahl Davis[8] and Wilkes and Churchill[9] were successful with $Pr \geq 0{\cdot}1$. Elder was able to obtain convergence with $Pr = 0{\cdot}01$, but only for small Ra.

The problem with central differences is that the resulting coefficient matrix contains off-diagonal values that are large relative to the diagonal values. Thus, using iteration methods becomes difficult.

One way to avoid large off-diagonal elements is to use the upwind method developed by Greenspan,[13] Schultz[1] and MacGregor and Emery.[14] Schultz obtained convergence for Ra up to $100\,000$ and Pr as small as $0{\cdot}00001$. However, since the method is only first-order, very small mesh sizes are needed to guarantee accuracy. De Vahl Davis and Jones[15] have published a comparison paper which summarizes results from 36 sources for the case of $Pr = 0{\cdot}71$, with Ra ranging from 10^3 to 10^6. They include one method which incorporates the second-order difference approximation with a fourth-order deferred correction step with mesh refinement and the bench mark solution by de Vahl Davis which uses the usual second-order difference approximation with a Richardson extrapolation technique. Shay and Schultz[12] presented a second-order method and a fourth-order method using extrapolation which compared favourably with the results of de Vahl Davis and Jones.[15] Saitoh and Hirose[16] use conventional five-point fourth-order approximations for the first and second derivatives to obtain a fourth-order method. The disadvantage here is that problems arise near the boundary. They also use scaled grid spacing with some new transformation function. Dennis and Hudson[17] developed a compact nine-point difference scheme which is fourth-order accurate. This method is a two-dimensional version of the methods of exponential type which uses the Numerov approximation. They obtained results up to $Ra=10^5$. It would be interesting to see if their accuracy is maintained for $Ra=10^6$.

In this paper we develop a fourth order difference method without using deferred correction or extrapolation. The method uses a compact nine-point stencil and is simple to implement, for it can use SOR iteration. The method converges for both large Ra and small Pr and can be extended to other partial differential equations. The method does not use the upwinding technique.

DIFFERENCE EQUATIONS

On and near the boundary

Since there is no explicit boundary condition for ω, we need to update boundary vorticities. Since $\omega=-\psi_{xx}$ on AD and BC and $\omega=-\psi_{yy}$ on AB and CD, from equation (1), we approximate boundary vorticities by setting

$$\psi_{xx}|_0=\sum_{i=0}^{5}\alpha_i\psi_i,$$

using the notation in Figure 2. If we expand each ψ_i about the point 0 in Figure 2, equate the coefficients and solve the linear system, we obtain

$$\begin{aligned}\omega_0&=-\psi_{xx}|_0\\&=\frac{-1}{60h^2}(225\psi_0-770\psi_1+1070\psi_2-780\psi_3+305\psi_4-50\psi_5)+O(h^4).\end{aligned}\tag{8}$$

The same formula is used for $\omega=-\psi_{yy}$ on AB and CD.

For the case $T_x=0$ on the side boundaries, if we use the formula

$$T_x|_0=(-2T_{-1}-3T_0+6T_1-T_2)/(\pm 6h)+O(h^3),$$

where the notation in Figure 2 is used, and solve $T_x|_0=0$ for T_{-1}, we obtain the temperature on the outer boundary

$$T_{-1}=(-3T_0+6T_1-T_2)/2+O(h^4).\tag{9}$$

We also found that using an inner boundary on the stream equation prevented numerical instability. (The term 'inner boundary' denotes the set of all points that lie at a distance h from the boundary.[18]) We reason that this is because it forces the derivative condition in (5)–(7) to be satisfied. If we use the difference formula $\psi_x|_0=(-11\psi_0+18\psi_1-9\psi_2+2\psi_3)/(\pm 6h)+O(h^3)$, we obtain as was done in Shay and Schultz:[12]

$$\psi_1=\psi_2/2-\psi_3/9+O(h^4).\tag{10}$$

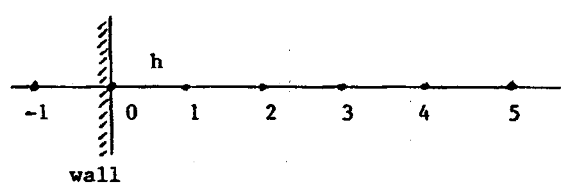

Figure 2

Or, if we use the difference formula $\psi_x|_0=(-25\psi_0+48\psi_1-36\psi_2+16\psi_3-3\psi_4)/(\pm 12h)+O(h^4)$, we obtain

$$\psi_1=3\psi_2/4-\psi_3/3+\psi_4/16+O(h^5). \tag{11}$$

Various higher- and lower-order approximations were tried in place of (8)–(11). However, the approximation (8), (9) and (11) gave the best overall results. The differences between results for the various approximations could be made negligible by using a small enough h.

In the interior region

In this section we formulate a stable fourth-order finite difference method that can solve equations (1)–(3), with a wide range of Pr and Ra values. Note that each of these equations is a special case of the elliptic equation

$$Lu\equiv u_{xx}+u_{yy}+p(x, y)u_x+q(x, y)u_y+r(x, y)u=s(x, y). \tag{12}$$

Since $r(x, y)\equiv 0$ in equations (1)–(3), we may write equation (12) as

$$Lu\equiv u_{xx}+u_{yy}+pu_x+qu_y=s, \tag{13}$$

where $p=p(x, y)$, $q=q(x, y)$ and $s=s(x, y)$. Note that $p=q=0$ and $s=-\omega$ in equation (1); $p=-\psi_y$, $q=\psi_x$ and $s=0$ in equation (2); $p=-(1/Pr)\psi_y$, $q=(1/Pr)\psi_x$ and $s=-RaT_y$ in equation (3).

To set up a finite difference equation for (13), we use Figure 3 which denotes the placement of nine points. We denote the point (x_i, y_j) as 0, (x_{i+1}, y_j) as 1, etc. Thus, we denote $u(x_i, y_j)=u_0$, $u(x_{i+1}, y_j)=u_1, \ldots, u(x_{i+1}, y_{j-1})=u_8$, and that $p(x_i, y_j)=p_0$, $q(x_i, y_j)=q_0$ and $s(x_i, y_j)=s_0$.

Using the above notation we obtain the following approximations for u_{xx}, u_{yy}, u_x and u_y at the point (x_i, y_j), numbered 0 in Figure 3:

$$u_{xx}|_0=\frac{u_3-2u_0+u_1}{h^2}-\frac{h^2}{12}u_{xxxx}|_0+O(h^4), \tag{14}$$

$$u_{yy}|_0=\frac{u_4-2u_0+u_2}{h^2}-\frac{h^2}{12}u_{yyyy}|_0+O(h^4), \tag{15}$$

$$u_x|_0=\frac{-u_3+u_1}{2h}-\frac{h^2}{6}u_{xxx}|_0+O(h^4), \tag{16}$$

$$u_y|_0=\frac{-u_4+u_2}{2h}-\frac{h^2}{6}u_{yyy}|_0+O(h^4). \tag{17}$$

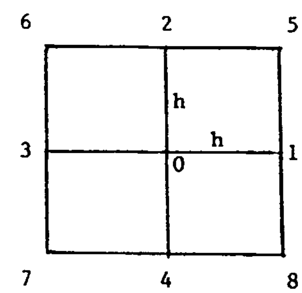

Figure 3

Now, substituting (14)–(17) into the differential equation (13), we obtain the central difference operator L_{hk}, defined by

$$L_h u_0 \equiv \sum_{i=0}^{4} \alpha_i u_i = s_0 + E_0[u], \tag{18}$$

where $E_0[u]$ is the truncation error and

$$\begin{aligned} \alpha_0 &= -\frac{4}{h^2}, \\ \alpha_1 &= \frac{1}{h^2} + \frac{p_0}{2h}, \\ \alpha_2 &= \frac{1}{h^2} + \frac{q_0}{2h}, \\ \alpha_3 &= \frac{1}{h^2} - \frac{p_0}{2h}, \\ \alpha_4 &= \frac{1}{h^2} - \frac{q_0}{2h}. \end{aligned} \tag{19}$$

Note that when $p(x, y)$ or $q(x, y)$ is large, that is, when Pr is very small, the central coefficient α_0 is small relative to the other coefficients, which is the main cause of the instability of the central difference method.

To obtain a stable fourth-order operator we rewrite the central difference approximation (18) in a form that includes the error terms. That is,

$$L_h u_0 - E_0[u] = s_0, \tag{20}$$

where

$$E_0[u] = \frac{h^2}{12} (u_{xxxx} + 2pu_{xxx} + u_{yyyy} + 2qu_{yyy})|_0 + O(h^4). \tag{21}$$

Then, we proceed to convert $E_0[u]$ into a combination of the lower-order derivatives u_{xx}, u_{yy}, u_x, u_y, u_{xy}, etc., which can be nicely approximated by nine points, as given in Figure 3, with a stabilizing effect. From (13), we have

$$u_{xx} = -(pu_x + qu_y + u_{yy}) + s.$$

Differentiating both sides of this equation with respect to x, we have

$$u_{xxx} = -(pu_{xx} + p_x u_x + qu_{yx} + q_x u_y + u_{yyx}) + s_x,$$

and

$$u_{xxxx} = -(pu_{xxx} + 2p_x u_{xx} + p_{xx} u_x + q_{xx} u_y + 2q_x u_{yx} + qu_{yxx} + u_{yyxx}) + s_{xx}.$$

Combining these equations, we have

$$\begin{aligned} u_{xxxx} + 2pu_{xxx} &= -(-pu_{xxx} + 2p_x u_{xx} + p_{xx} u_x + q_{xx} u_y + 2q_x u_{yx} + qu_{yxx} + u_{yyxx}) + s_{xx} \\ &= -[(p^2 + 2p_x)u_{xx} + (pp_x + p_{xx})u_x + (pq_x + q_{xx})u_y \\ &\quad + (pq + 2q_x)u_{yx} + pu_{yyx} + qu_{yxx} + u_{yyxx}] + ps_x + s_{xx}. \end{aligned} \tag{22}$$

Similarly, we have

$$u_{yyyy}+2qu_{yyy}=-[(q^2+2q_y)u_{yy}+(qq_y+q_{yy})u_y+(qp_y+p_{yy})u_x \\ +(pq+2p_y)u_{xy}+qu_{xxy}+pu_{xyy}+u_{xxyy}]+qs_y+s_{yy}. \tag{23}$$

Then, substituting (22) and (23) into (21) and assuming that $u_{xy}=u_{yx}$, $u_{xxy}=u_{yxx}$ and $u_{xxyy}=u_{yyxx}$, we have

$$E_0[u]=-\frac{h^2}{12}[(p^2+2p_x)u_{xx}+(pp_x+qp_y+p_{xx}+p_{yy})u_x+(q^2+2q_y)u_{yy}+(pq_x+qq_y+q_{xx}+q_{yy})u_y \\ +2(pq+p_y+q_x)u_{xy}+2(pu_{xyy}+qu_{xxy}+u_{xxyy})]|_0+\frac{h^2}{12}(ps_x+qs_y+s_{xx}+s_{yy})|_0+O(h^4). \tag{24}$$

Note that the terms $(p^2+2p_x)u_{xx}$ and $(q^2+2q_y)u_{yy}$ can produce strong central coefficients, when they are approximated by (14) and (15).

Using (24) in (20) and rearranging (20), we obtain

$$L_h u_0-\tilde{E}_0[u]=s_0^*, \tag{25}$$

where

$$\tilde{E}_0[u]=E_0[u]-\frac{h^2}{12}(ps_x+qs_y+s_{xx}+s_{yy})|_0, \tag{26}$$

and

$$s_0^*=s_0+\frac{h^2}{12}(ps_x+qs_y+s_{xx}+s_{yy})|_0. \tag{27}$$

To approximate $\tilde{E}_0[u]$, we need to approximate u_{xy}, u_{xxy}, u_{yyx} and u_{xxyy}. So we derive the following formulas by Taylor series expansions:

$$u_{xy}|_0=\frac{1}{4h^2}(u_7-u_6-u_8+u_5)+O(h^2), \tag{28}$$

$$u_{xxy}|_0=\frac{1}{2h^3}(-u_7+2u_4-u_8+u_6-2u_2+u_5)+O(h^2), \tag{29}$$

$$u_{yyx}|_0=\frac{1}{2h^3}(-u_7+2u_3-u_6+u_8-2u_1+u_5)+O(h^2), \tag{30}$$

$$u_{xxyy}|_0=\frac{1}{h^4}(u_7-2u_4+u_8-2u_3+4u_0-2u_1+u_6-2u_2+u_5)+O(h^2). \tag{31}$$

Note that we can derive the formula (28) by $u_{xy}|_0=(-u_x|_4+u_x|_2)/(2h)$, (29) by $u_{xxy}|_0=(-u_{xx}|_4+u_{xx}|_2)/(2h)$, (30) by $u_{yyx}|_0=(-u_{yy}|_3+u_{yy}|_1)/(2h)$ and (31) by $u_{xxyy}|_0=(u_{xx}|_4-2u_{xx}|_0+u_{xx}|_2)/h^2$. Although the order of the error term for each formula is not obvious, it is easy to prove second-order accuracy by Taylor expansion.

Now, we approximate $\tilde{E}_0[u]$ using the formulas (14)–(17) and (28)–(31)

$$\tilde{E}_0[u]=\sum_{i=0}^{8}\beta_i u_i+O(h^4), \tag{32}$$

where

$$\begin{aligned}
\beta_0 &= -\frac{4}{6h^2}+\frac{1}{6}(p^2+q^2+2p_x+2q_y)|_0,\\
\beta_1 &= \frac{2}{6h^2}+\frac{p_0}{6h}-\frac{1}{12}(p^2+2p_x)|_0-\frac{h}{24}(pp_x+qp_y+p_{xx}+p_{yy})|_0,\\
\beta_2 &= \frac{2}{6h^2}+\frac{q_0}{6h}-\frac{1}{12}(q^2+2q_y)|_0-\frac{h}{24}(pq_x+qq_y+q_{xx}+q_{yy})|_0,\\
\beta_3 &= \frac{2}{6h^2}-\frac{p_0}{6h}-\frac{1}{12}(p^2+2p_x)|_0+\frac{h}{24}(pp_x+qp_y+p_{xx}+p_{yy})|_0,\\
\beta_4 &= \frac{2}{6h^2}-\frac{q_0}{6h}-\frac{1}{12}(q^2+2q_y)|_0+\frac{h}{24}(pq_x+qq_y+q_{xx}+q_{yy})|_0,\\
\beta_5 &= \frac{-1}{6h^2}-\frac{p_0+q_0}{12h}-\frac{(pq+p_y+q_x)|_0}{24},\\
\beta_6 &= \frac{-1}{6h^2}-\frac{-p_0+q_0}{12h}+\frac{(pq+p_y+q_x)|_0}{24},\\
\beta_7 &= \frac{-1}{6h^2}-\frac{-p_0-q_0}{12h}-\frac{(pq+p_y+q_x)|_0}{24},\\
\beta_8 &= \frac{-1}{6h^2}-\frac{p_0-q_0}{12h}+\frac{(pq+p_y+q_x)|_0}{24}.
\end{aligned} \tag{33}$$

Finally, substituting (32) into (25), we obtain the stable fourth-order operator L_h^* for equation (13), defined by

$$L_h^* u_0 \equiv \sum_{i=0}^{8} \alpha_i^* u_i = s_0^* + E_0^*[u], \tag{34}$$

where s_0^* is given by (27) and $\alpha_i^* = \alpha_i - \beta_i$. Also, we see that $E_0^*[u]$, the local truncation error, is $O(h^4)$.

Now, we are ready to set up the difference equations for each of the equations (1)–(3). First, for equation (1), where $p=q=0$ and $s=-\omega$, we have the following difference equation:

$$L_h^* \psi_0 \equiv \sum_{i=0}^{8} \alpha_i^* \psi_i = s_0^* + E_0^*[\psi], \tag{35}$$

where

$$\begin{aligned}
\alpha_0^* &= -\frac{20}{6h^2},\\
\alpha_1^* = \alpha_2^* = \alpha_3^* = \alpha_4^* &= \frac{4}{6h^2},\\
\alpha_5^* = \alpha_6^* = \alpha_7^* = \alpha_8^* &= \frac{1}{6h^2},
\end{aligned} \tag{36}$$

and

$$s_0^* = -\omega_0 - \frac{h^2}{12}(\omega_{xx} + \omega_{yy})|_0 . \tag{37}$$

Note that if $s = -\omega$ is harmonic, $s_0^* = s_0 = -\omega_0$, since $\omega_{xx} + \omega_{yy} = 0$ in (37). This implies that, in such case, the operator L_h^* is identical to the nine-point formula for the Poisson equation, which has $O(h^6)$ accuracy as discussed in Reference 19 (p. 195). When $s = -\omega$ is not harmonic, we can approximate s_0^* in (37) with $O(h^4)$ accuracy by approximating ω_{xx} and ω_{yy} with central difference formulas (14) and (15). Since the local truncation error $E_0^*[\psi] = O(h^4)$, as was the case for $E_0^*[u]$ in (34), L_h^* has fourth-order accuracy in this case. We remark that our fourth-order formula (35)–(37) for this particular case is identical to the fourth-order formula for the Poisson equation with non-harmonic non-homogeneous terms, derived in Reference 20 (p. 282) in a different way.

Next, for equation (2), where $p = -\psi_y$, $q = \psi_x$ and $s = 0$, we have the following difference equation:

$$L_h^* T_0 \equiv \sum_{i=0}^{8} \alpha_i^* T_i = E_0^*[T], \tag{38}$$

where

$$\begin{aligned}
\alpha_0^* &= -\frac{20}{6h^2} - \frac{1}{6}(p^2 + q^2 + 2p_x + 2q_y)|_0 ,\\
\alpha_1^* &= \frac{4}{6h^2} + \frac{p_0}{3h} + \frac{1}{12}(p^2 + 2p_x)|_0 + \frac{h}{24}(pp_x + qp_y + p_{xx} + p_{yy})|_0 ,\\
\alpha_2^* &= \frac{4}{6h^2} + \frac{q_0}{3h} + \frac{1}{12}(q^2 + 2q_y)|_0 + \frac{h}{24}(pq_x + qq_y + q_{xx} + q_{yy})|_0 ,\\
\alpha_3^* &= \frac{4}{6h^2} - \frac{p_0}{3h} + \frac{1}{12}(p^2 + 2p_x)|_0 - \frac{h}{24}(pp_x + qp_y + p_{xx} + p_{yy})|_0 ,\\
\alpha_4^* &= \frac{4}{6h^2} - \frac{q_0}{3h} + \frac{1}{12}(q^2 + 2q_y)|_0 - \frac{h}{24}(pq_x + qq_y + q_{xx} + q_{yy})|_0 ,\\
\alpha_5^* &= \frac{1}{6h^2} + \frac{p_0 + q_0}{12h} + \frac{(pq + p_y + q_x)|_0}{24} ,\\
\alpha_6^* &= \frac{1}{6h^2} + \frac{-p_0 + q_0}{12h} - \frac{(pq + p_y + q_x)|_0}{24} ,\\
\alpha_7^* &= \frac{1}{6h^2} + \frac{-p_0 - q_0}{12h} + \frac{(pq + p_y + q_x)|_0}{24} ,\\
\alpha_8^* &= \frac{1}{6h^2} + \frac{p_0 - q_0}{12h} - \frac{(pq + p_y + q_x)|_0}{24} .
\end{aligned} \tag{39}$$

In (39), $p = -\psi_y$ and $q = \psi_x$, from which we have $p_x = -\psi_{xy}$, $p_y = -\psi_{yy}$, $q_x = \psi_{xx}$ and $q_y = \psi_{xy}$. We also have $p_{xx} + p_{yy} = -\psi_{xxy} - \psi_{yyy} = -(\partial/\partial y)(\psi_{xx} + \psi_{yy}) = \omega_y$, using equation (1) and, similarly, $q_{xx} + q_{yy} = -\omega_x$.

For a fully fourth-order difference scheme we need to approximate $p = -\psi_y$ and $q = \psi_x$ with

$O(h^4)$ accuracy, which is done as follows:

$$\psi_y = (-\psi_4 + \psi_2)/(2h) - h^2 \psi_{yyy}/6 + O(h^4)$$
$$= (-\psi_4 + \psi_2)/(2h) + h^2(\psi_{xxy} + \omega_y)/6 + O(h^4),$$

since $\psi_{yyy} = -(\psi_{xxy} + \omega_y)$ from equation (1). In the similar way,

$$\psi_x = (-\psi_3 + \psi_1)/(2h) + h^2(\psi_{yyx} + \omega_x)/6 + O(h^4).$$

We then approximate ψ_{xxy}, ψ_{yyx}, ω_x and ω_y with $O(h^2)$ accuracy using (29), (30), (16) and (17), respectively. However, note that for p_x, p_y, $p_{xx}+p_{yy}$, etc. $O(h^2)$ accurate approximations are enough, since they are parts of (21) and (26). Also note, that $E_0^*[T] = O(h^4)$ as was the case for $E_0^*[u]$ in (34). We, thus, have a fully fourth-order difference scheme for equation (2).

Finally, for equation (3), where $p = -(1/Pr)\psi_y$, $q = (1/Pr)\psi_x$ and $s = -RaT_y$, we have the following difference equation:

$$L_h^* \omega_0 \equiv \sum_{i=0}^{8} \alpha_i^* \omega_i = s_0^* + E_0^*[\omega], \tag{40}$$

where

$$\alpha_0^* = -\frac{20}{6h^2} - \frac{1}{6}(p^2 + q^2 + 2p_x + 2q_y)|_0,$$

$$\alpha_1^* = \frac{4}{6h^2} + \frac{p_0}{3h} + \frac{1}{12}(p^2 + 2p_x)|_0 + \frac{h}{24}(pp_x + qp_y + p_{xx} + p_{yy})|_0,$$

$$\alpha_2^* = \frac{4}{6h^2} + \frac{q_0}{3h} + \frac{1}{12}(q^2 + 2q_y)|_0 + \frac{h}{24}(pq_x + qq_y + q_{xx} + q_{yy})|_0,$$

$$\alpha_3^* = \frac{4}{6h^2} - \frac{p_0}{3h} + \frac{1}{12}(p^2 + 2p_x)|_0 - \frac{h}{24}(pp_x + qp_y + p_{xx} + p_{yy})|_0,$$

$$\alpha_4^* = \frac{4}{6h^2} - \frac{q_0}{3h} + \frac{1}{12}(q^2 + 2q_y)|_0 - \frac{h}{24}(pq_x + qq_y + q_{xx} + q_{yy})|_0, \tag{41}$$

$$\alpha_5^* = \frac{1}{6h^2} + \frac{p_0 + q_0}{12h} + \frac{(pq + p_y + q_x)|_0}{24},$$

$$\alpha_6^* = \frac{1}{6h^2} + \frac{-p_0 + q_0}{12h} - \frac{(pq + p_y + q_x)|_0}{24},$$

$$\alpha_7^* = \frac{1}{6h^2} + \frac{-p_0 - q_0}{12h} + \frac{(pq + p_y + q_x)|_0}{24},$$

$$\alpha_8^* = \frac{1}{6h^2} + \frac{p_0 - q_0}{12h} - \frac{(pq + p_y + q_x)|_0}{24}.$$

In (41), $p = -(1/Pr)\psi_y$ and $q = (1/Pr)\psi_x$, from which we have $p_x = -(1/Pr)\psi_{xy}$, $p_y = -(1/Pr)\psi_{yy}$, $q_x = (1/Pr)\psi_{xx}$ and $q_y = (1/Pr)\psi_{xy}$. We also have, using equation (1),

$$p_{xx} + p_{yy} = -(1/Pr)(\psi_{xxy} + \psi_{yyy}) = -(1/Pr)\frac{\partial}{\partial y}(\psi_{xx} + \psi_{yy}) = (1/Pr)\omega_y.$$

Similarly, we have

$$q_{xx}+q_{yy}=-(1/Pr)\omega_x.$$

All these terms are approximated as was done before in (39).

And s_0^* is, from (27),

$$s_0^*=s_0+\frac{h^2}{12}(ps_x+qs_y+s_{xx}+s_{yy})|_0, \tag{42}$$

where $p=-(1/Pr)\psi_y$, $q=(1/Pr)\psi_x$ and $s=-RaT_y$. From $s=-RaT_y$, we have $s_x=-RaT_{xy}$, $s_y=-RaT_{yy}$. Then, using equation (2), we have

$$s_{xx}+s_{yy}=-Ra(T_{xxy}+T_{yyy})=-Ra\frac{\partial}{\partial y}(T_{xx}+T_{yy})=-Ra\frac{\partial}{\partial y}(-\psi_x T_y+\psi_y T_x),$$

from which it follows that

$$\begin{aligned}s_{xx}+s_{yy}&=-Ra(-\psi_{xy}T_y-\psi_x T_{yy}+\psi_{yy}T_x+\psi_y T_{xy})\\&=-Ra\cdot Pr(p_x T_y-qT_{yy}-p_y T_x-pT_{xy}).\end{aligned}$$

Note that s_0^* will be $O(h^4)$ accurate when we approximate $s=-RaT_y$ with $O(h^4)$ accuracy and all other terms with $O(h^2)$ accuracy. Therefore, we approximate $s=-RaT_y$ as follows:

$$\begin{aligned}s=-RaT_y&=-Ra(-T_4+T_2)/(2h)+Rah^2T_{yyy}/6+O(h^4)\\&=-Ra(-T_4+T_2)/(2h)-Rah^2(T_{xxy}+\psi_{xy}T_y+\psi_x T_{yy}-\psi_{yy}T_x-\psi_y T_{xy})/6+O(h^4),\end{aligned}$$

since $T_{yyy}=-(\partial/\partial y)(T_{xx}+\psi_x T_y-\psi_y T_x)$ from equation (2). Also, $E_0^*[\omega]=O(h^4)$ as was the case for $E_0^*[u]$ in (34).

RESULTS AND CONCLUSIONS

Solutions have been obtained for $1\leq Ra\leq 10^6$ and Prandtl numbers $0{\cdot}00001\leq Pr\leq 10$. Results have been obtained for the number of grid spacings n in each direction varying from 10 to 120. All results were run on the IBM RISC 6000 series model 530. The time for $Ra=10^4$ and $n=20$ was 15 s, for $Ra=10^4$ and $n=40$ was 3 min 50 s and for $Ra=10^4$ and $n=80$ was 60 min 27 s. We used SOR iteration with an absolute convergence test of 0·000002. As in References 12 and 15, the average Nusselt number Nu was calculated through the use of a three-point approximation to T_y at the cold wall and Simpson's rule to approximate $\int_0^1 \partial T/\partial y|_{y=0}\,dx$.

Figures 4–15 contain level curves for the stream, vorticity and temperature functions.

Tables I–VI contain results which are compared with those of de Vahl Davis and Jones,[15] Shay and Schultz,[12] Dennis and Hudson[17] and Saitoh and Hirose.[16]

Table I compares the results from the first-order method of Schultz,[1] the second-order method of Shay and Schultz[12] and the current fourth-order method for the case where $T=y$ on the side boundaries. Note that the results from this paper for $n=10$ are better than the results for $n=80$ in Reference 1 and close to the results of $n=40$ in Reference 12. Note also that the results for the fourth-order method are virtually identical for $n=40$, 60 and 80, showing excellent convergence.

Tables II and III compare the results from the current fourth-order method and the second-order method in Reference 12 for the case $T_x=0$ on the side boundaries for various values of h for $Ra=10^5$. Note that the results for $n=40$ in this paper are close to the extrapolated results from Reference 12. For example, we have $\psi_{mid}=9{\cdot}093$ for $n=40$, while the extrapolated result from Reference 12 is 9·089 using $n=40$ and $n=80$.

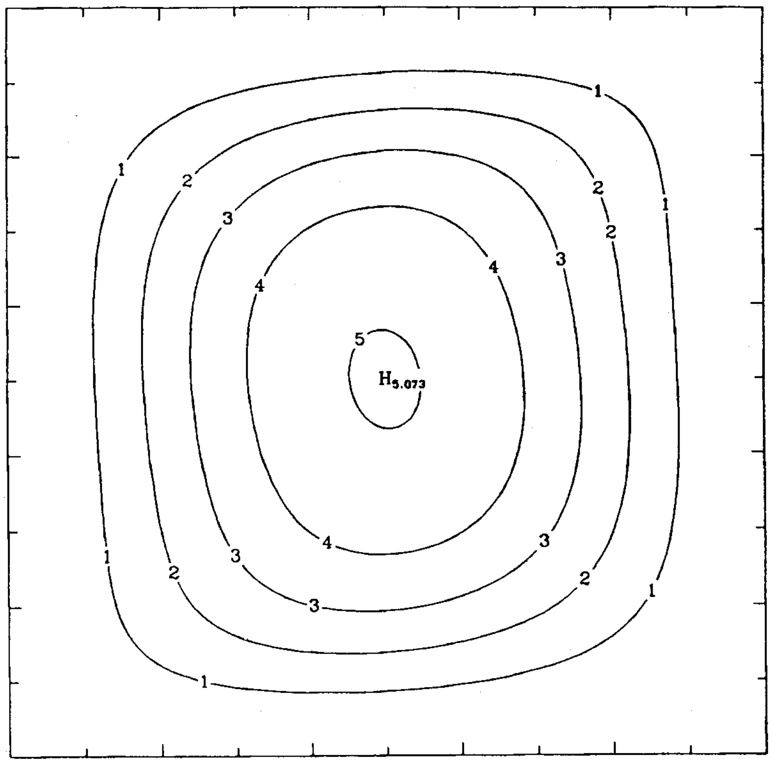

Figure 4. Streamlines for $Ra = 10^4$ and $Pr = 0{\cdot}71$, 60×60 grid

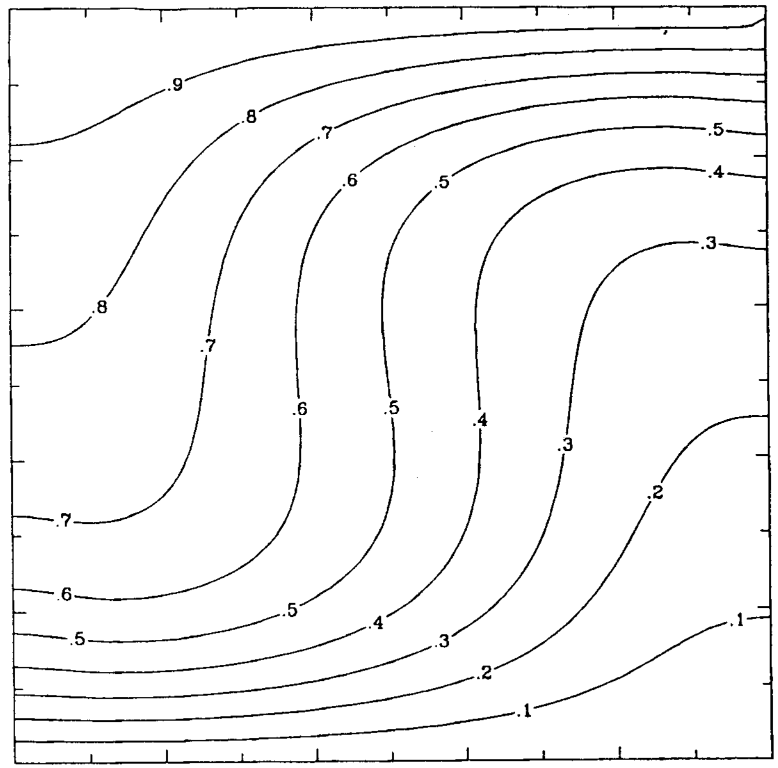

Figure 5. Temperature for $Ra = 10^4$ and $Pr = 0{\cdot}71$, 60×60 grid

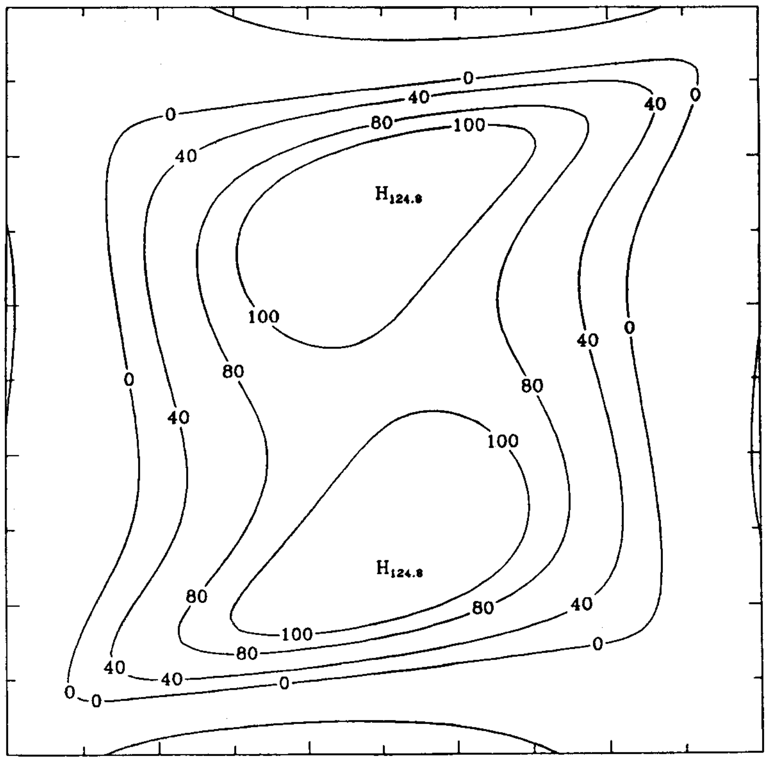

Figure 6. Vorticities for $Ra=10^4$ and $Pr=0{\cdot}71$, 60×60 grid

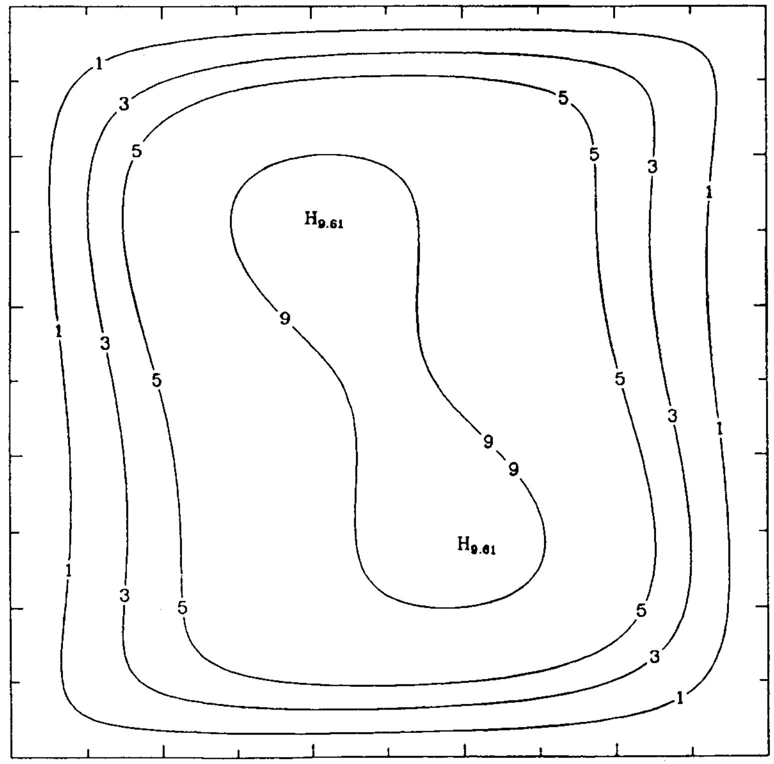

Figure 7. Streamlines for $Ra=10^5$ and $Pr=0{\cdot}71$, 60×60 grid

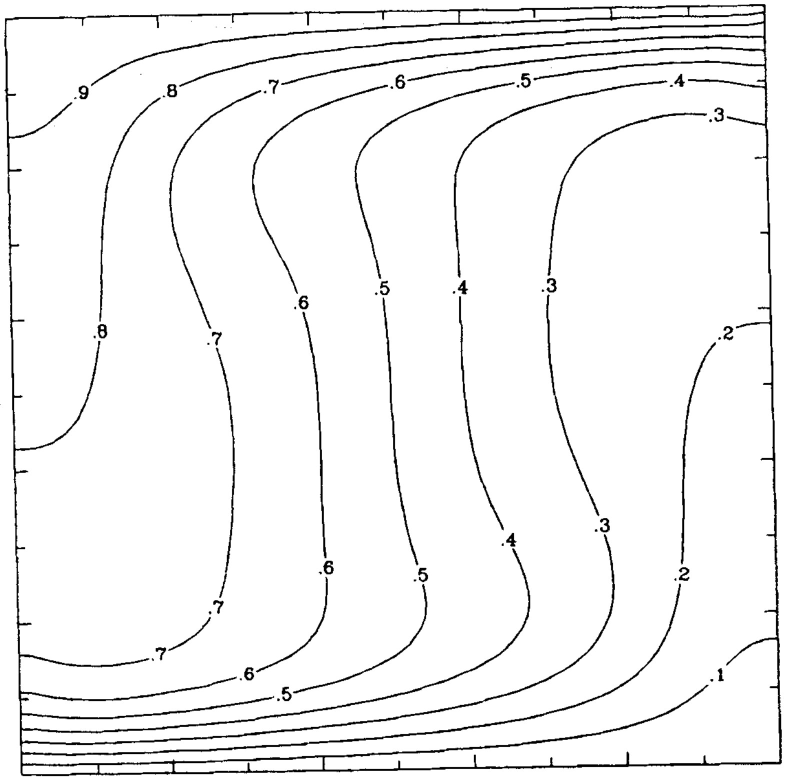

Figure 8. Temperature for $Ra = 10^5$ and $Pr = 0{\cdot}71$, 60×60 grid

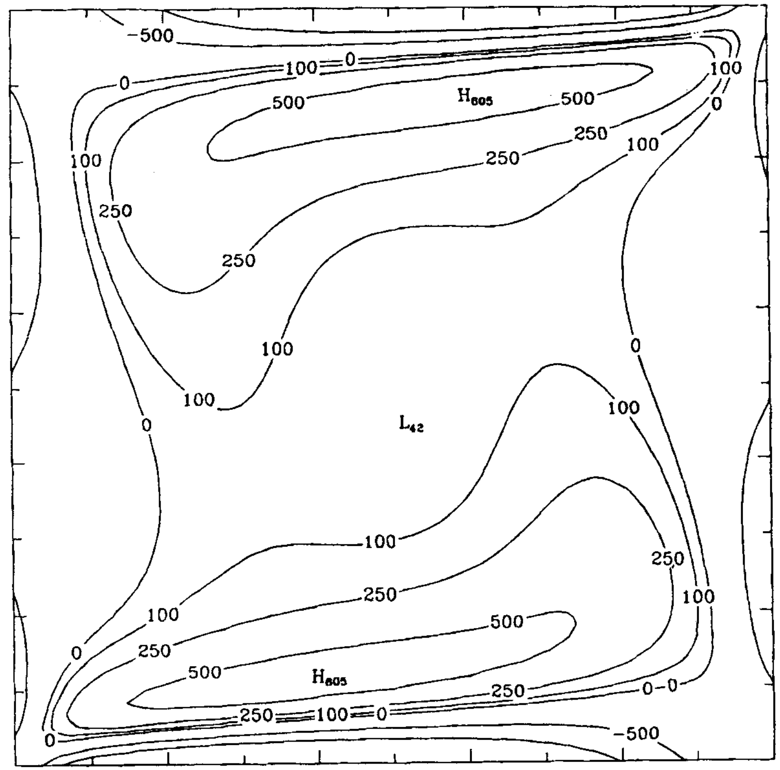

Figure 9. Vorticities for $Ra = 10^5$ and $Pr = 0{\cdot}71$, 60×60 grid

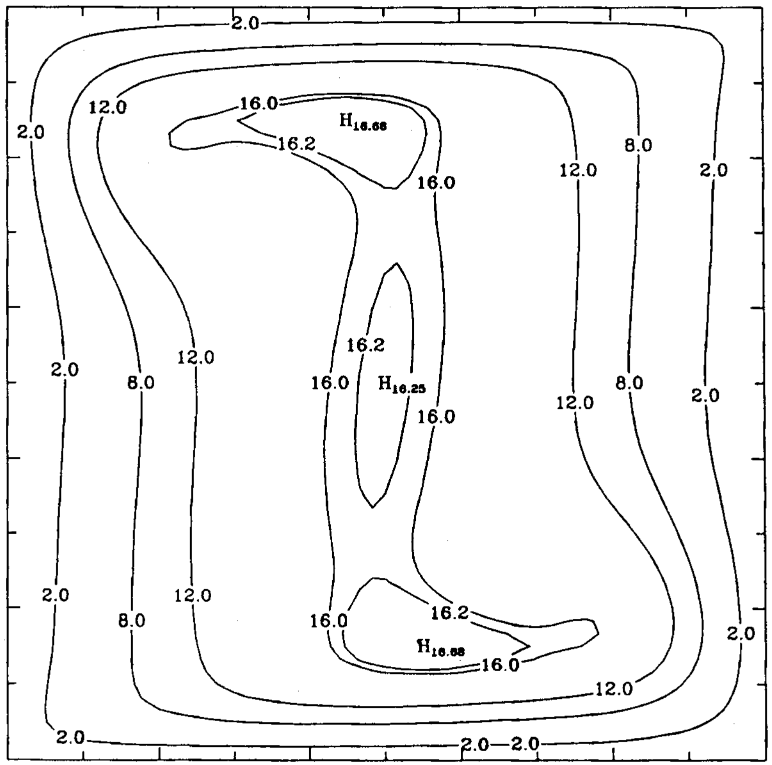

Figure 10. Streamlines for $Ra = 10^6$ and $Pr = 0{\cdot}71$, 60×60 grid

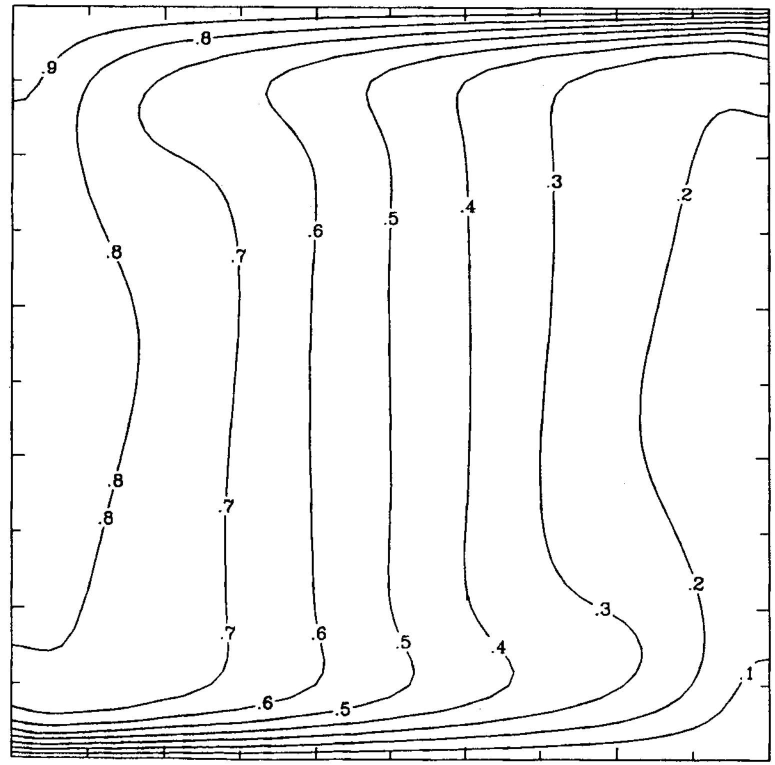

Figure 11. Temperature for $Ra = 10^6$ and $Pr = 0{\cdot}71$, 60×60 grid

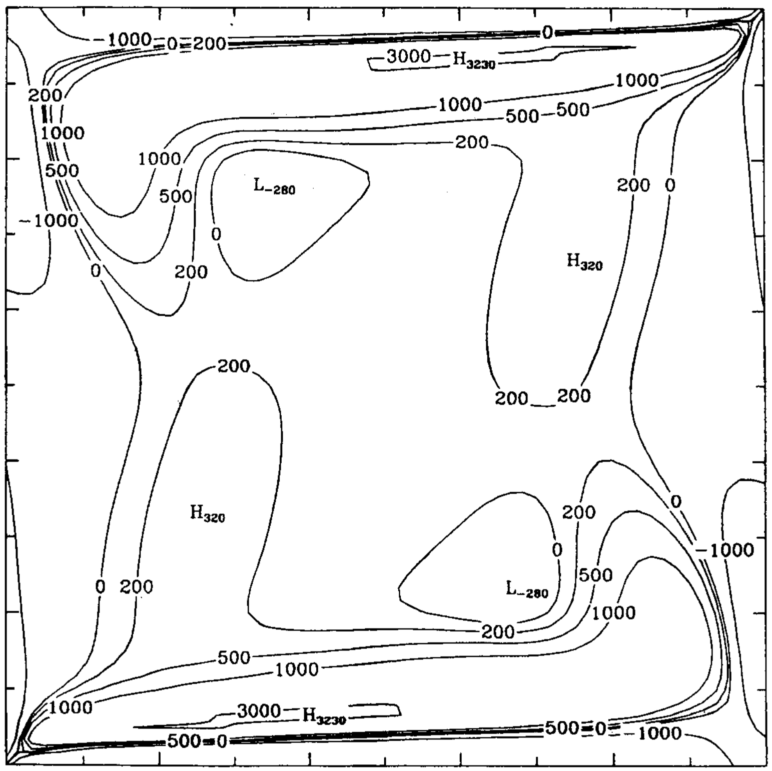

Figure 12. Vorticities for $Ra = 10^6$ and $Pr = 0{\cdot}71$, 60×60 grid

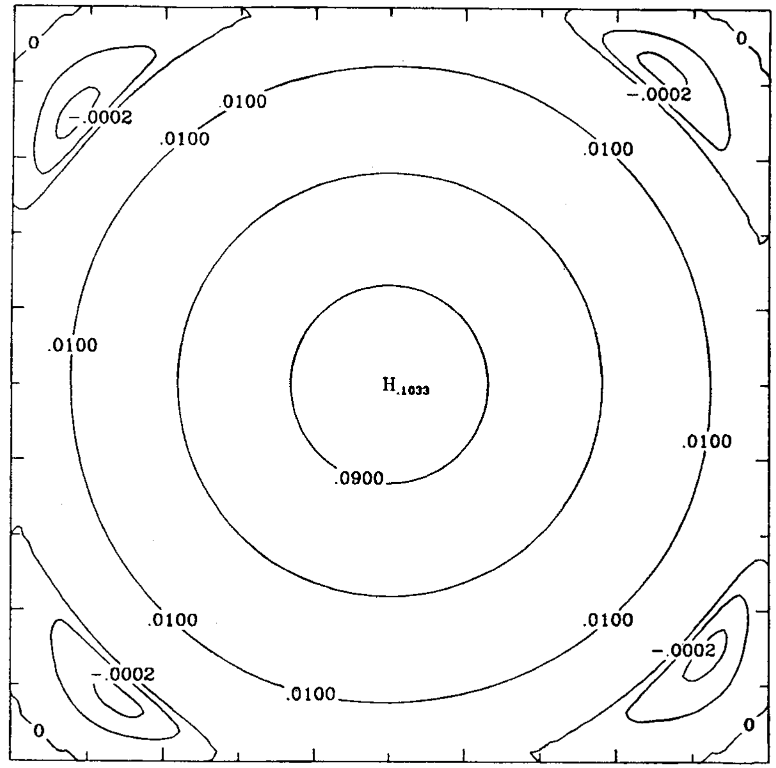

Figure 13. Streamlines for $Ra = 100$ and $Pr = 0{\cdot}0001$, 80×80 grid

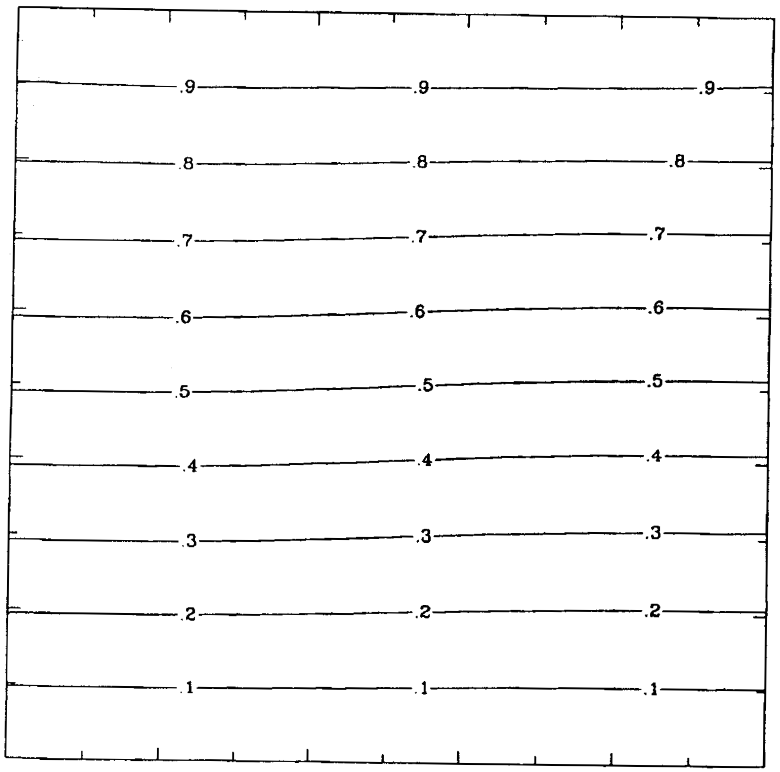

Figure 14. Temperature for $Ra = 100$ and $Pr = 0{\cdot}0001$, 80×80 grid

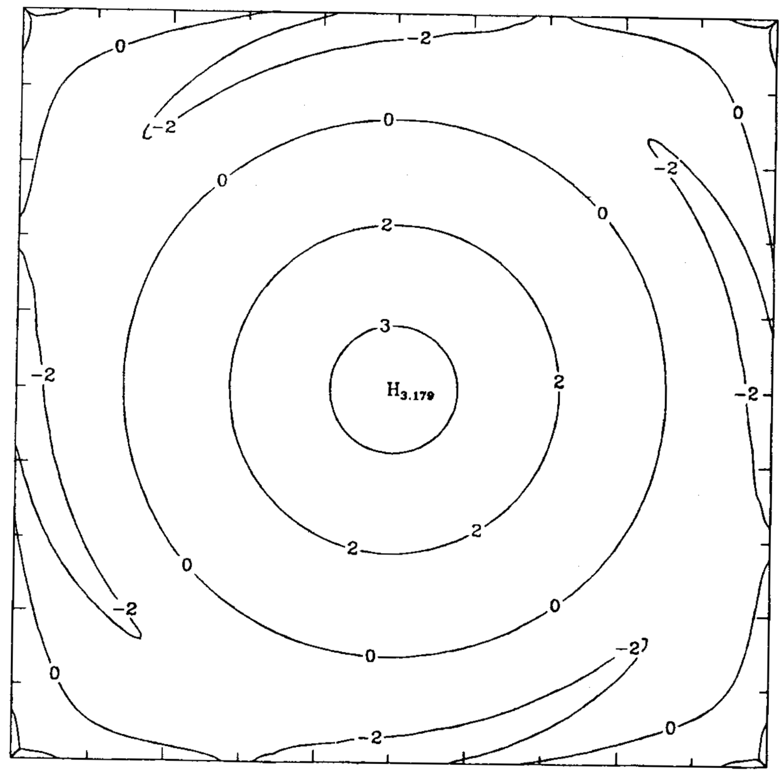

Figure 15. Vorticities for $Ra = 100$ and $Pr = 0{\cdot}0001$, 80×80 grid

Table I. Comparison of results from the current fourth-order method and the first-order method in Reference 1 and the second-order method in Reference 12. $Ra=10^4$ and $Pr=0{\cdot}73$ and $T=y$ on the side boundaries

	First-order method Schultz[1]		Second-order method Shay and Schultz[12]		Fourth-order method Current study	
n	ψ_{mid}	ω_{mid}	ψ_{mid}	ω_{mid}	ψ_{mid}	ω_{mid}
10	7·962	168·0	6·764	139·4	6·243	129·60
20	7·066	142·0	6·430	130·7	6·324	129·01
40	6·590	135·0	6·357	129·4	6·3430	129·07
60	n.a.	n.a.	n.a.	n.a.	6·3436	129·08
80	6·423	132·3	n.a.	n.a.	6·3437	129·08

n.a. = Not available.

Table II. Results from the second-order method in Reference 12 for $Ra=10^5$ and $Pr=0{\cdot}71$ as a function of n. Extrapolated results are obtained from the results with $n=40$ and $n=80$

	$n=20$	$n=40$	$n=80$	Extrapolated
Nu	4·943	4·658	4·543	4·505
ψ_{mid}	9·31	9·24	9·127	9·089
ψ_{max}	9·83	9·74	9·628	9·591

Table III. Results from the current fourth-order method for $Ra=10^5$ and $Pr=0{\cdot}71$ as a function of n

	$n=20$	$n=40$	$n=60$	$n=80$	$n=100$
Nu	4·730	4·580	4·542	4·529	4·524
ψ_{mid}	8·988	9·093	9·113	9·116	9·116
ψ_{max}	9·450	9·588	9·616	9·618	9·617

Table IV compares the present results for $Ra=10^3$, 10^4, 10^5 and 10^6 with those in Shay and Schultz,[12] the bench mark solutions in de Vahl Davis,[15] the compact difference solutions in Dennis and Hudson[17] and the results from Saitoh and Hirose.[16] The table shows the excellent agreement of the present results with those of the bench mark solution for all the values of Ra from 10^3 to 10^6, with those of Saitoh and Hirose for $Ra=10^4$ and $Ra=10^6$ and with those of the compact difference method up to $Ra=10^5$. (The results of Saitoh and Hirose are not available for $Ra=10^3$ and $Ra=10^6$ and those of the compact difference method are not available for $Ra=10^6$.) The difference is well within 0·2% up to $Ra=10^5$, except with one result from the compact difference scheme, and within 0·8% for $Ra=10^6$.

Table V shows the convergence of the current fourth-order method as n increases (h decreases). Note that there is very little change in the results in Table V for $n=40$ and 60 for $Ra\leq 10^4$, and for $n=80$ and 100 for $Ra=10^5$, showing excellent convergence. For $Ra=10^6$ the results are quite

Table IV. Comparison of the best results from Shay and Schultz,[12] the bench mark solutions by de Vahl Davis,[15] the compact difference solutions by Dennis and Hudson[17] and the results from Saitoh and Hirose[16] ($Pr=0{\cdot}71$)

Reference	Ra	10^3	10^4	10^5	10^6
Shay and Schultz[12]	Nu	n.a.	2·257	4·505	n.a.
	ψ_{mid}	n.a.	5·070	9·089	n.a.
	ψ_{max}	n.a.	5·070	9·591	n.a.
de Vahl Davis and Jones[15]	Nu	1·118	2·243	4·519	8·800
	ψ_{mid}	1·174	5·071	9·111	16·32
	ψ_{max}	n.a.	n.a.	9·612	16·750
Saitoh and Hirose[16]	Nu	n.a.	2·2424	n.a.	8·7126
	ψ_{mid}	n.a.	5·0731	n.a.	16·245
	ψ_{max}	n.a.	n.a.	n.a.	n.a.
Dennis and Hudson[17]	Nu	1·1176	2·2396	4·4959	n.a.
	ψ_{mid}	1·1747	5·0735	9·1126	n.a.
	ψ_{max}	n.a.	n.a.	n.a.	n.a.
Present method	Nu	1·116	2·243	4·524	8·870
	ψ_{mid}	1·174	5·073	9·116	16·379
	ψ_{max}	1·174	5·073	9·617	16·804

Table V. Comparison of the present fourth-order results as a function of Ra and n ($Pr=0{\cdot}71$)

		$n=10$	$n=20$	$n=40$	$n=60$	$n=80$
$Ra=10^3$	Nu	1·109	1·112	1·115	1·116	1·116
	ψ_{mid}	1·162	1·171	1·173	1·174	1·174
	ψ_{max}	1·162	1·171	1·173	1·174	1·174
		$n=10$	$n=20$	$n=40$	$n=60$	$n=80$
$Ra=10^4$	Nu	2·327	2·265	2·245	2·243	2·243
	ψ_{mid}	5·050	5·065	5·074	5·073	5·073
	ψ_{max}	5·050	5·065	5·074	5·073	5·073
		$n=20$	$n=40$	$n=60$	$n=80$	$n=100$
$Ra=10^5$	Nu	4·730	4·580	4·542	4·529	4·524
	ψ_{mid}	8·988	9·093	9·113	9·116	9·116
	ψ_{max}	9·450	9·588	9·616	9·618	9·617
		$n=40$	$n=60$	$n=80$	$n=100$	$n=120$
$Ra=10^6$	Nu	9·292	9·059	8·951	8·898	8·870
	ψ_{mid}	16·069	16·253	16·337	16·368	16·379
	ψ_{max}	16·485	16·679	16·763	16·793	16·804

Table VI. Comparison of results from the current fourth-order method and the second-order method in Shay and Schultz[12] for $Ra = 100$ and $Pr = 0.0001$

Method	n	ψ_{mid}	Nu
Second order[12]	80	0·1054	1·00045
Fourth order	80	0·1033	0·9995

Table VII. Results from the current fourth-order method for $Ra = 100$ and $Pr = 10$ as a function of n

	$n = 10$	$n = 20$	$n = 40$	$n = 60$	$n = 80$
Nu	0·9903	0·9958	0·9982	0·9988	0·9980
ψ_{mid}	0·1248	0·1259	0·1262	0·1263	0·1263
ω_{mid}	3·494	3·508	3·515	3·517	3·517

close for $n = 80$, 100 and 120. Although the present method showed excellent convergence, a finer mesh size was needed for larger Ra to obtain accuracy. This limitation may be overcome by the use of mesh refinement or co-ordinate transformation, which is under consideration.

In addition, a collection of results from 36 sources is summarized in Reference 15. One source used a mesh size $n = 80$ for $Ra = 10^6$, but had difficulties preserving the symmetry of the problem. Also, none of the methods in References 15, 16 and 17 indicates success with small Pr. But the present method produced results for a wide range of Pr. Table VI compares the results of the present method with those of Shay and Schultz[12] for $Pr = 0.0001$ and $Ra = 100$, and Table VII shows the results for $Pr = 10$ and $Ra = 100$. Our numerical results showed that for $Pr > 1$ the present method converged very nicely. The results did not show any change for $Pr > 10$, $Ra = 100$.

It is the success of the current method with the wide range of Ra, Pr and mesh sizes that indicates the potential of this method as an accurate and stable numerical method applicable to a wide range of problems.

REFERENCES

1. D. H. Schultz, 'Numerical solution for the flow of a fluid in a heated closed cavity', *Q. J. Mech. Appl. Math.*, **XXVI**, (Pt. 2), 173–192 (1973).
2. W. Mull and H. Reicher, *Gesundheitsingenieur*, 28 (1930).
3. J. W. Elder, 'Numerical experiments with free convection in a vertical slot', *J. Fluid Mech.*, **24**, 823–843 (1966).
4. K. E. Torrance, L. Orloff and J. A. Rockett, *J. Fluid Mech.*, **36**, 21 (1969).
5. E. R. G. Eckert and W. O. Carlson, *Int. J. Heat Mass Transfer*, **2**, 106 (1961).
6. G. Poots, 'Heat Transfer by laminar free convection in enclosed plane gas layers', *Q. J. Mech. Appl. Math.*, **11**, 257–273 (1958).
7. J. B. Rosen, 'Approximate solution to transient Navier–Stokes cavity convection problems', *Technical Report No. 32*, Dept. of Computer Science, University of Wisconsin, 1968.
8. G. de Vahl Davis, 'Laminar natural convection in an enclosed rectangular cavity', *Int. J. Heat Mass Transfer*, **11**, 1675–1693 (1968).
9. J. O. Wilkes and S. W. Churchill, 'The finite difference computation of natural convection in a rectangular enclosure', *AICHE*, **12**, 161–166 (1966).

10. A. Rubel and F. Landis, 'Numerical study of natural convection within rectangular enclosures', *Phys. Fluids* (Suppl. II), 208–213 (1969).
11. M. E. Newell and F. W. Schmidt, 'Heat transfer by laminar natural convection within rectangular enclosures', *J. Heat Transfer*, **92**, 159–168 (1970).
12. W. A. Shay and D. H. Schultz, 'A second-order approximation to natural convection for large Rayleigh numbers and small Prandtl numbers', *Int. j. numer. methods fluids*, **5**, 427–438 (1985).
13. D. Greenspan, *Discrete Numerical Methods in Physics and Engineering*, Academic Press, New York, 1974.
14. R. K. MacGregor and A. F. Emery, 'Free convection through vertical plane layers—moderate and high Prandtl number fluids', *J. Heat Transfer*, **91**, 391–402 (1969).
15. G. de Vahl Davis and I. P. Jones, 'Natural convection in a square cavity: a comparison exercise', *Int. j. numer. methods fluids*, **3**, 227–248 (1983).
16. T. Saitoh and K. Hirose, 'High-accuracy bench mark solutions to natural convection in a square cavity', *Comput. Mech.*, **4**, 417–427 (1989).
17. S. C. R. Dennis and J. D. Hudson, 'Compact h^4 finite-difference approximations to operators of Navier–Stokes type', *J. Comput. Phys.*, **85**, 390–416 (1989).
18. D. Greenspan, *Lectures on the Numerical Solution of Linear, Singular and Non-linear Differential Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1968.
19. G. E. Forsythe and W. R. Wasow, *Finite Difference Methods for Partial Differntial Equations*, Wiley, New York, 1960.
20. J. C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadswords & Books/Cole Advanced Books & Software, California, 1989.